

EXPLICIT TRACES OF FUNCTIONS ON SOBOLEV SPACES AND QUASI-OPTIMAL LINEAR INTERPOLATORS

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ABSTRACT. Let $\Lambda \subset \mathbb{R}$ be a strictly increasing sequence. For $r = 1, 2$, we give a simple explicit expression for an equivalent norm on the trace spaces $W_p^r(\mathbb{R})|_\Lambda$, $L_p^r(\mathbb{R})|_\Lambda$ of the non-homogeneous and homogeneous Sobolev spaces with r derivatives $W_p^r(\mathbb{R})$, $L_p^r(\mathbb{R})$. We also construct an interpolating spline of low degree having optimal norm up to a constant factor. A general result relating interpolation in $L_p^r(\mathbb{R})$ and $W_p^r(\mathbb{R})$ for all $r \geq 1$ is also given.

1. INTRODUCTION

Let $I \subset \mathbb{R}$ be an interval (possibly infinite). The Sobolev space $W_p^r(I)$ for integer $r \geq 1$ is defined as the completion of the space of complex functions $F \in \mathcal{C}^\infty(I)$ such that the norm

$$\|F\|_{W_p^r(I)}^p = \int_I |F(x)|^p dx + \int_I |F^{(r)}(x)|^p dx$$

is finite. By the Sobolev embedding theorem, if $F \in W_p^r$, $1 < p < \infty$, then $F^{(r-1)}$ is absolutely continuous. Similarly, one can define the homogeneous Sobolev space $L_p^r(I)$ as the space of functions of finite seminorm

$$\|F\|_{L_p^r(I)}^p = \int_I |F^{(r)}(x)|^p dx.$$

In fact, given any $x_0 \in I$, the expression

$$\left(|F(x_0)|^p + \dots + |F^{(r-1)}(x_0)|^p + \|F\|_{L_p^r(I)}^p \right)^{\frac{1}{p}}$$

gives a Banach-space norm on $L_p^r(I)$. If I is a finite interval, then $L_p^r(I) = W_p^r(I)$, and this Banach norm is equivalent to $\|\cdot\|_{W_p^r(I)}$.

In the sequel, let \mathbb{X} stand either for $W_p^r(I)$ or $L_p^r(I)$. If $\Lambda \subset I$ is any set, we can define the trace space

$$\mathbb{X}|_\Lambda = \{F|_\Lambda : F \in \mathbb{X}\}.$$

There is the following natural choice of a norm (or seminorm) in the trace space:

$$\|f\|_{\mathbb{X}|_\Lambda} = \inf\{\|F\|_{\mathbb{X}} : F|_\Lambda = f\} = \min\{\|F\|_{\mathbb{X}} : F|_\Lambda = f\}.$$

Here the minimum is attained because, for $1 < p < \infty$, the spaces $L_p^r(I)$ and $W_p^r(I)$ are uniformly convex.

Throughout this note we will restrict ourselves to the case where $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ is an increasing sequence:

$$\lambda_n < \lambda_{n+1}, \quad n \in \mathbb{Z}.$$

As is known [4, 12], the following questions are of relevance in the study of the trace space.

Problem 1. Given some $f : \Lambda \rightarrow \mathbb{C}$, when does f extend to a function $F \in \mathbb{X}$ satisfying $F|_\Lambda = f$? Can one take this extension to depend linearly on the data f ?

Problem 2. Find an explicit simple formula for a norm (or seminorm) equivalent to $\|\cdot\|_{\mathbb{X}|_\Lambda}$.

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Observe that in our case, $\mathbb{X}|_{\Lambda}$ is a sequence space. Hence, we are searching for an intrinsic characterization of the sequences in $\mathbb{X}|_{\Lambda}$. The solution of Problem 2 can help us solve Problem 1, because if we have a formula for an equivalent norm in the trace space, we can characterize the trace space as the space of sequences having finite norm. This does not address the question of the linear dependence of the extension. To solve this, it is usual to consider the next problem.

Problem 3. Does there exist a linear operator $T : \mathbb{X}|_{\Lambda} \rightarrow \mathbb{X}$ which is bounded and satisfies $(Tf)|_{\Lambda} = f$, for all $f \in \mathbb{X}|_{\Lambda}$? If so, can one give a simple formula for one such T ?

An operator having these properties is usually called a bounded linear extension operator. In the case when Λ is a sequence, the problem of finding an F such that $F|_{\Lambda} = f$ can be understood as an interpolation problem (we say that such an F interpolates the data f). A bounded linear extension operator provides a function Tf interpolating the data f and satisfying

$$(1) \quad \|f\|_{\mathbb{X}|_{\Lambda}} \leq \|Tf\|_{\mathbb{X}} \leq \|T\| \|f\|_{\mathbb{X}|_{\Lambda}}.$$

Here the first inequality comes from the definition of the trace norm. Hence, a bounded linear operator gives an interpolating function Tf having optimal norm up to a constant factor. Because of this property, we will refer to these operators as quasi-optimal interpolators. In applications, it is important to find interpolators T with small norm.

The construction of a quasi-optimal interpolator given by a simple formula can also allow us to solve Problem 2, as (1) shows that if we put $\|f\|_{\text{eq}} = \|Tf\|_{\mathbb{X}}$, then $\|\cdot\|_{\text{eq}}$ is an equivalent norm to $\|\cdot\|_{\mathbb{X}|_{\Lambda}}$.

In this note we solve Problems 1–3 for the Sobolev spaces $W_p^r(I)$, $L_p^r(I)$ for $r = 1, 2$; $1 < p < \infty$. We use the standard notation $f(x_1, \dots, x_n)$ for the divided differences:

$$\begin{aligned} f(x_1, x_2) &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \\ f(x_1, \dots, x_n) &= \frac{f(x_2, \dots, x_n) - f(x_1, \dots, x_{n-1})}{x_n - x_1}, \quad n \geq 3. \end{aligned}$$

We will always assume that

$$(2) \quad I = \bigcup_{n \in \mathbb{Z}} [\lambda_n, \lambda_{n+1}].$$

This is a general enough case. To see this, let $I = (\alpha, \beta)$ and let (a, b) be the right hand side of (2). Assume, for instance, that $a = \alpha$, $b < \beta$ and that we are dealing with the space $L_p^r(I)$ (the other cases are very similar). Observe that $b < +\infty$, so that $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ clusters at b . Assume that $F \in L_p^r[a, b]$ is such that $F|_{\Lambda} = f$. Using the fact that $F^{(j)}$, $0 \leq j < r$, are continuous, and the mean value theorem for divided differences (see (14)), one sees that the values of $F(b), \dots, F^{(r-1)}(b)$ are completely determined by f . Hence, to find a quasi-optimal interpolating function in $L_p^r(I)$, we first find a quasi-optimal interpolating function $F \in L_p^r[a, b]$. Then we have to extend this F to a quasi-optimal $\tilde{F} \in L_p^r(I)$. Thus, we need to define $\tilde{F}|_{(b, \beta)}$, preserving the continuity of $\tilde{F}^{(j)}$, $0 \leq j < r$. Since the values $\tilde{F}(b), \dots, \tilde{F}^{(r-1)}(b)$ are fixed by f , one can even construct $G = \tilde{F}|_{(b, \beta)}$ beforehand. Moreover, finding a quasi-optimal $G \in L_p^r(b, \beta)$ with given $G(b), \dots, G^{(r-1)}(b)$ is an easy problem.

Define

$$(3) \quad \begin{aligned} \|f\|_{\text{eq}, L}^p &= \sum_{n \in \mathbb{Z}} (\lambda_{n+r} - \lambda_n) |f(\lambda_n, \dots, \lambda_{n+r})|^p, \\ \|f\|_{\text{eq}, W}^p &= \|f\|_{\text{eq}, L}^p + \sum_{j=0}^{r-1} \sum_{n \in \mathbb{Z}} (\lambda_{n+r} - \lambda_n)^{jp+1} |f(\lambda_n, \dots, \lambda_{n+j})|^p. \end{aligned}$$

In this paper we show that

$$(4) \quad C(r, p) \|f\|_{\text{eq}, L} \leq \|f\|_{L_p^r(I)|_{\Lambda}} \leq C'(r, p) \|f\|_{\text{eq}, L}, \quad r = 1, 2, \quad 1 < p < \infty.$$

Let

$$h_n = \lambda_{n+1} - \lambda_n, \quad n \in \mathbb{Z}$$

be the interpolation steps. We say that the steps are uniformly bounded if there is a constant K such that

$$(5) \quad h_n \leq K.$$

Then we also show that if (5) holds, then

$$(6) \quad C(r, p, K) \|f\|_{\text{eq}, W} \leq \|f\|_{W_p^r(I)|_\Lambda} \leq C'(r, p, K) \|f\|_{\text{eq}, W}, \quad r = 1, 2, \quad 1 < p < \infty.$$

In fact, we prove a general result (for any $r \geq 1$) which relates quasi-optimal interpolation in L_p^r and W_p^r (see Theorem 2). Using this Theorem, (6) will follow from (4).

We conjecture that (4) and (6) are also true for $r \geq 3$. See Section 6 for a discussion of this and some other open questions.

In the case $r = 2$, the term $j = 1$ in the expression for $\|f\|_{\text{eq}, W}$ can be eliminated, thus giving a simpler expression. We will show in Proposition 6 that for $r = 2$, $\|f\|_{\text{eq}, W}^p$ is equivalent to

$$(7) \quad \|f\|_{\text{simp}, W}^p = \|f\|_{\text{eq}, L}^p + \sum_{n \in \mathbb{Z}} (\lambda_{n+1} - \lambda_{n-1}) |f(\lambda_n)|^p.$$

For $r \geq 3$, in general, one cannot hope to remove all the terms $1 \leq j \leq r-1$ from the expression for $\|f\|_{\text{eq}, W}$, thus obtaining a expression similar to (7). However, if the interval I is large enough in comparison with K , this can be done. See Section 5 for these results.

Spline interpolators are a useful class of interpolators because they provide computationally simple formulas. We will say that T is a spline interpolator of degree d if there is a decomposition of I into closed intervals $\{I_j\}$ which intersect only at their endpoints, and such that for any f , the restriction of Tf to each interval I_j is a polynomial of degree at most d . In this paper we construct quasi-optimal spline interpolators Φ_1, Φ_2 (for $r = 1, 2$ respectively) given by simple expressions.

Extension by smooth functions dates back to H. Whitney [17]. In 1934, he solved Problem 1 for the space $C^m(\mathbb{R})$. Brudnyi and Shvartsman extended Whitney's results to the space $C^{1,\omega}(\mathbb{R}^n)$ in [3]. Fefferman made more progress in [5, 6], where he solved Problem 1 for the spaces $C^m(\mathbb{R}^n)$ and $C^{m,1}(\mathbb{R}^n)$. He also proved the existence of bounded linear extension operators. Recently, G.K. Luli generalized these results to $C^{m,\omega}(\mathbb{R}^n)$ in [14].

The concept of the depth of an extension operator appears in some of these works. A linear extension operator $T : \mathbb{X}|_\Lambda \rightarrow \mathbb{X}$ is said to be of bounded depth if there is some $D \geq 0$ such that

$$(Tf)(x) = \sum_{y \in \Lambda} \phi(x, y) f(y),$$

and for each x , $\phi(x, y)$ is nonzero for at most D different values of y . If T has bounded depth, its depth is the smallest integer D such that this holds.

It is easy to see that the spline interpolators constructed in this paper have bounded depth. In fact, Φ_1 has depth 2 and Φ_2 has depth 3.

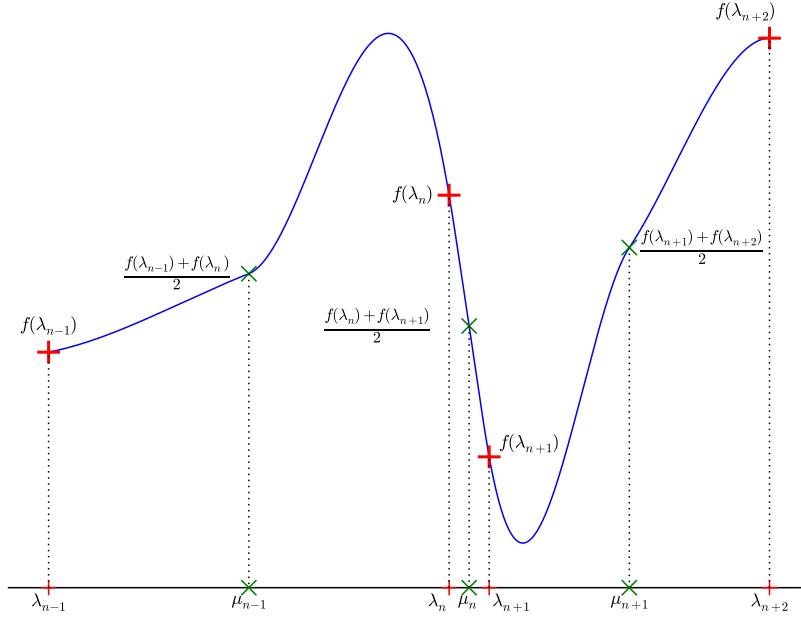
B. Klartag and Fefferman have addressed the problem of computing efficiently the extension function and the norm in the trace space in [10, 11]. See the expository paper [7]. However, their algorithm does not give explicit simple formulas.

In [13], G.K. Luli constructs a bounded depth interpolator in $L_p^r(\mathbb{R})$ by pasting interpolating polynomials using partitions of unity. A. Israel gives a result for $L_p^2(\mathbb{R}^2)$ in [12]. Shvartsman has obtained in [15, 16] results for the non-homogeneous Sobolev space W_p^1 , both in \mathbb{R}^n and in metric spaces.

In a recent paper [8], Fefferman, Israel and Luli extend Israel's result to $L_p^r(\mathbb{R}^n)$. They show that a linear extension operator can be constructed such that it has assisted bounded depth. In general, one cannot hope to construct extension operators of uniformly bounded depth in L_p^r , as they show in [9].

We remark that although our setting is less general than those mentioned above, we will obtain a very simple expression for the norm in the trace space and explicitly construct a simple quasi-optimal interpolator. We believe that these results can be of interest in numerical analysis.

In many numerical methods, such as the finite element methods and the Galerkin methods, an approximation for the solution of an equation in an infinite dimensional function space is

FIGURE 1. Graph of $\Phi_2 f$.

searched in some finite dimensional subspace. The linear spaces of quasi-optimal splines we construct (corresponding to a finite number of interpolation nodes) may be good candidates for these finite dimensional spaces.

2. THE DEFINITION OF THE INTERPOLATORS Φ_1 , Φ_2

2.1. Definition of Φ_1 . The interpolator Φ_1 is just the piecewise linear interpolator. This means that $\Phi_1 f$ is an affine function in each interval $[\lambda_n, \lambda_{n+1}]$. Together with the condition $(\Phi_1 f)(\lambda_n) = f(\lambda_n)$, this completely determines Φ_1 .

The interpolator Φ_1 is given by the following formula:

$$(8) \quad (\Phi_1 f)(x) = f(\lambda_n) \frac{\lambda_{n+1} - x}{h_n} + f(\lambda_{n+1}) \frac{x - \lambda_n}{h_n}, \quad \lambda_n \leq x \leq \lambda_{n+1}.$$

2.2. Definition of Φ_2 . The interpolator Φ_2 is a spline interpolator of degree 3. It is defined as follows.

Let

$$\mu_n = \frac{\lambda_n + \lambda_{n+1}}{2}.$$

Given $f : \Lambda \rightarrow \mathbb{C}$, we construct cubic polynomials on each of the intervals $[\mu_{n-1}, \lambda_n]$ and $[\lambda_n, \mu_n]$ such that the resulting cubic spline $\Phi_2 f$ is \mathcal{C}^1 and satisfies $(\Phi_2 f)(\lambda_n) = f(\lambda_n)$.

Put

$$\alpha_n(f) = \frac{h_n f(\lambda_{n-1}, \lambda_n) + h_{n-1} f(\lambda_n, \lambda_{n+1})}{h_{n-1} + h_n}.$$

The conditions

$$(9) \quad \begin{aligned} (\Phi_2 f)(\lambda_n) &= f(\lambda_n), & (\Phi_2 f)'(\lambda_n) &= \alpha_n(f) \\ (\Phi_2 f)(\mu_n) &= \frac{f(\lambda_n) + f(\lambda_{n+1})}{2}, & (\Phi_2 f)'(\mu_n) &= f(\lambda_n, \lambda_{n+1}). \end{aligned}$$

determine $\Phi_2 f$, since $\Phi_2 f$ is a piecewise cubic polynomial on each of the intervals $[\mu_{n-1}, \lambda_n]$, $[\lambda_n, \mu_n]$. Observe that, restricted to the interval $[\mu_{n-1}, \mu_n]$, $\Phi_2 f$ only depends on $f(\lambda_{n-1})$, $f(\lambda_n)$, $f(\lambda_{n+1})$.

See Figure 1 for a typical graph of $\Phi_2 f$. The interpolation nodes $(\lambda_n, f(\lambda_n))$ are shown as +'s and the auxiliary nodes $(\mu_n, (f(\lambda_n) + f(\lambda_{n+1}))/2)$ are shown as 'x's.

We also have the following explicit formula for $\Phi_2 f$. Define

$$q(x) = 4(x^2 - x^3)$$

and observe that q satisfies the boundary conditions

$$(10) \quad q(0) = q'(0) = 0, \quad q\left(\frac{1}{2}\right) = \frac{1}{2}, \quad q'\left(\frac{1}{2}\right) = 1.$$

Then we have

$$(11) \quad \begin{aligned} (\Phi_2 f)(x) &= f(\lambda_n) + \alpha_n(f)(x - \lambda_n) + h_{n-1}^2 f(\lambda_{n-1}, \lambda_n, \lambda_{n+1}) q\left(\frac{\lambda_n - x}{h_{n-1}}\right), \quad \mu_{n-1} \leq x \leq \lambda_n, \\ (\Phi_2 f)(x) &= f(\lambda_n) + \alpha_n(f)(x - \lambda_n) + h_n^2 f(\lambda_{n-1}, \lambda_n, \lambda_{n+1}) q\left(\frac{x - \lambda_n}{h_n}\right), \quad \lambda_n \leq x \leq \mu_n. \end{aligned}$$

By using the identities

$$\begin{aligned} h_{n-1}^2 f(\lambda_{n-1}, \lambda_n, \lambda_{n+1}) &= f(\lambda_{n-1}) - f(\lambda_n) + h_{n-1} \alpha_n(f), \\ h_n^2 f(\lambda_{n-1}, \lambda_n, \lambda_{n+1}) &= f(\lambda_{n+1}) - f(\lambda_n) - h_n \alpha_n(f), \end{aligned}$$

and (10), it is easy to see that (11) defines the same spline as (9).

3. MAIN RESULTS

In the sequel, we will use the notation $\|f\|_1 \approx \|f\|_2$, which means that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, i.e., there are constants $C > 0$ and $C' < \infty$ such that, for all f ,

$$C\|f\|_1 \leq \|f\|_2 \leq C'\|f\|_1.$$

The first result solves Problems 1–3 for $L_p^r(I)$, $W_p^r(I)$, $r = 1, 2$.

Theorem 1. *Let $r = 1, 2$, and $1 < p < \infty$. Define $\|f\|_{eq,W}^p$, $\|f\|_{eq,L}^p$ from (3). Then the following statements are true:*

(a) *The seminorm $\|\cdot\|_{eq,L}$ gives an equivalent seminorm in $L_p^r(I)$:*

$$\|f\|_{L_p^r(I)|_\Lambda} \approx \|f\|_{eq,L},$$

where the equivalence constants depend only on r, p . Moreover, $\Phi_r : L_p^r(I)|_\Lambda \rightarrow L_p^r(I)$ is quasi-optimal and $\|\Phi_r\| \leq C(r, p)$.

(b) *If (5) holds, then the norm $\|\cdot\|_{eq,W}$ gives an equivalent norm in $W_p^r(I)$:*

$$\|f\|_{W_p^r(I)|_\Lambda} \approx \|f\|_{eq,W}.$$

The equivalence constants depend only on r, p, K . Moreover, $\Phi_r : W_p^r(I)|_\Lambda \rightarrow W_p^r(I)$ is quasi-optimal and $\|\Phi_r\| \leq C'(r, p, K)$.

If the assumption (5) in this Theorem is not fulfilled, we can always use the following trick. First fix some $L > 0$. Let $\{J_k\}$ be the collection of all the intervals $J_k = [\lambda_k, \lambda_{k+1}]$ such that $h_k > L$. Given a function f on Λ , proceeding as if (5) were true, one obtains an interpolating function F on I . However, $\|F\|_{W_p^r(I)}$ is not comparable with $\|f\|_{W_p^r(I)|_\Lambda}$.

Now one chooses $C^\infty(\mathbb{R})$ functions φ_k such that $\|\varphi_k\|_{C^r(\mathbb{R})} \leq M$, $0 \leq \varphi_k(x) \leq 1$, $\varphi_k(x) = 1$ if $x \notin \mathring{J}_k$ and $|\text{supp}(\varphi_k|_{J_k})| \leq L$. Put $\varphi = \prod \varphi_k$. Then one can see that $\|\varphi F\|_{W_p^r(I)}$ will be comparable with $\|f\|_{W_p^r(I)|_\Lambda}$.

The second result relates quasi-optimal interpolation on L_p^r and W_p^r . It will be used to deduce part (b) of the Theorem above from part (a). It could also be of help in attacking the problem for $r \geq 3$.

Theorem 2. *Let $r \geq 1$ be arbitrary and $1 < p < \infty$. Suppose that (5) holds. The following statements are true:*

(a) Assume that the following equivalence of norms holds:

$$(12) \quad \|f\|_{eq,L} \approx \|f\|_{L_p^r(I)|_\Lambda}, \quad f \in L_p^r(I)|_\Lambda.$$

Then the following equivalence of norms also holds:

$$\|f\|_{eq,W} \approx \|f\|_{W_p^r(I)|_\Lambda}, \quad f \in W_p^r(I)|_\Lambda.$$

The equivalence constants depend only on r, p, K and the equivalence constants in (12).

(b) If $T : L_p^r(I)|_\Lambda \rightarrow L_p^r(I)$ is a quasi-optimal interpolator, then $T : W_p^r(I)|_\Lambda \rightarrow W_p^r(I)$ is also a quasi-optimal interpolator.

The proofs of Theorems 1 and 2 will be given at the end of the next section.

4. PROOF OF THE RESULTS

We start by proving that

$$(13) \quad \|f\|_{eq,X} \leq C \|f\|_{X_p^r(I)|_\Lambda}, \quad X = L, W.$$

This will follow from a few easy estimates and is valid for all $r \geq 1$.

Lemma 3. Let $r \geq 1$, and $J \subset \mathbb{R}$ be a closed interval with $|J| \leq K$. Let $x_1, \dots, x_{k+1} \in J$, $0 \leq k \leq r-1$. Then, for any $F \in W_p^r(J)$,

$$\|F\|_{W_p^r(J)}^p \geq C|J|^{kp+1}|F(x_1, \dots, x_{k+1})|^p,$$

where $C > 0$ is a constant $C = C(r, p, K)$.

Proof. By the mean value theorem for divided differences (see, for instance, [1, Chapter 3.2]), there is some $\xi \in J$ such that

$$(14) \quad F^{(k)}(\xi) = \frac{F(x_1, \dots, x_{k+1})}{k!}.$$

It is an easy consequence of the Sobolev embedding theorem that there is some constant $C_0 > 0$ depending only on r, p such that if $G \in W_p^r[0, 1]$ and $\eta \in [0, 1]$, then

$$\|G\|_{W_p^r[0,1]}^p \geq C_0|G^{(k)}(\eta)|^p.$$

Let $\varphi(x)$ be the affine transformation taking $[0, 1]$ to J . If $F \in W_p^r(J)$, put $G(x) = F(\varphi(x))$ and compute

$$\begin{aligned} \frac{1}{|J|} \|F\|_{W_p^r(J)}^p &= \frac{1}{|J|} \int_J (|F(x)|^p + |F^{(r)}(x)|^p) dx = \int_0^1 (|G(x)|^p + |J|^{-rp}|G^{(r)}(x)|^p) dx \\ &\geq C_1 \|G\|_{W_p^r[0,1]}^p \geq C_1 C_0 |G^{(k)}(\varphi^{-1}(\xi))|^p = C_1 C_0 |J|^{kp} |F^{(k)}(\xi)|^p \\ &\geq C_1 C_0 [(r-1)!]^{-p} |J|^{kp} |F(x_0, \dots, x_{k+1})|^p \end{aligned}$$

where $C_1 = \min\{1, K^{-rp}\}$. The lemma holds with $C = C_1 C_0 [(r-1)!]^{-p}$. \square

Let us now recall the following result from [13].

Lemma A (see [13]). Let $r \geq 1$, $x_1 < x_2 < \dots < x_{r+1}$, $J = [x_1, x_{r+1}]$, $F \in L_p^r(J)$. Then

$$|F(x_1, x_2, \dots, x_{r+1})|^p |J| \leq [(r-1)!]^{-p} \|F\|_{L_p^r(J)}^p.$$

With these two Lemmas, now we can prove the following Proposition, which will give (13).

Proposition 4. Let $r \geq 1$. Put $f = F|_\Lambda$ and define $\|f\|_{eq,L}$ and $\|f\|_{eq,W}$ from (3). The following statements hold:

- (a) If $F \in L_p^r(I)$, then $\|F\|_{L_p^r(I)} \geq C \|f\|_{eq,L}$ for some constant $C > 0$ depending only on r, p .
- (b) If (5) holds and $F \in W_p^r(I)$, then $\|F\|_{W_p^r(I)} \geq C' \|f\|_{eq,W}$ for some constant $C' > 0$ depending only on r, p, K .

Proof. If $F \in L_p^r(I)$, we have

$$\|F\|_{L_p^r(I)}^p = \frac{1}{r} \sum_{n \in \mathbb{Z}} \|F\|_{L_p^r[\lambda_n, \lambda_{n+r}]}^p \geq \frac{[(r-1)!]^p}{r} \sum_{n \in \mathbb{Z}} (\lambda_{n+r} - \lambda_n) |F(\lambda_n, \dots, \lambda_{n+r})|^p,$$

by Lemma A. This implies (a), with $C = (r-1)!/r^{1/p}$.

To prove (b), fix $k = 0, 1, \dots, r-1$ and use Lemma 3 to obtain

$$\|F\|_{W_p^r(I)}^p = \frac{1}{r} \sum_{n \in \mathbb{Z}} \|F\|_{W_p^r[\lambda_n, \lambda_{n+r}]}^p \geq \frac{C''}{r} \sum_{n \in \mathbb{Z}} (\lambda_{n+r} - \lambda_n)^{kp+1} |F(\lambda_n, \dots, \lambda_{n+k})|^p.$$

By summing these inequalities over $k = 0, \dots, r-1$ together with (a), we obtain (b). \square

Our goal now is to prove Theorem 2. We will need the following Lemma, which is related to Friedrichs' inequality.

Lemma 5. *Let $J \subset \mathbb{R}$ be an interval of finite length. If $F \in L_p^r(J)$ and $\xi_0, \dots, \xi_{r-1} \in J$, then*

$$\|F\|_{L_p^r(J)}^p \leq C \left[|J|^{rp} \|F\|_{L_p^r(J)}^p + \sum_{j=0}^{r-1} |J|^{jp+1} |F^{(j)}(\xi_j)|^p \right],$$

for some constant $C = C(r, p)$.

Proof. We have, for $j = 0, \dots, r-1$,

$$F^{(j)}(x) = F^{(j)}(\xi_j) + \int_{\xi_j}^x F^{(j+1)}(t) dt,$$

so that

$$|F^{(j)}(x)| \leq |F^{(j)}(\xi_j)| + \int_J |F^{(j+1)}(t)| dt.$$

Repeated application of this inequality yields

$$\begin{aligned} |F(x)| &\leq |F(\xi_0)| + \int_J |F'(t)| dt \leq |F(\xi_0)| + |J| |F'(\xi_1)| + |J| \int_J |F''(t)| dt \\ &\leq \dots \leq \sum_{j=0}^{r-1} |J|^j |F^{(j)}(\xi_j)| + |J|^{r-1} \int_J |F^{(r)}(t)| dt \leq \sum_{j=0}^{r-1} |J|^j |F^{(j)}(\xi_j)| + |J|^{r-1+\frac{p-1}{p}} \|F\|_{L_p^r(J)}^p, \end{aligned}$$

where the last inequality comes from Hölder's inequality. Using Hölder's inequality for sums, we see that

$$|F(x)|^p \leq (r+1)^{p-1} \left[\sum_{j=0}^{r-1} |J|^{jp} |F^{(j)}(\xi_j)|^p + |J|^{rp-1} \|F\|_{L_p^r(J)}^p \right].$$

Now the lemma follows by integrating this inequality. \square

Now we can give the proof of Theorem 2.

Proof of Theorem 2. Let $F \in W_p^r(I)$ and fix $n \in \mathbb{Z}$. By the mean value theorem for divided differences, we can choose $\xi_0, \dots, \xi_{r-1} \in [\lambda_n, \lambda_{n+r}]$ such that

$$F^{(j)}(\xi_j) = \frac{F(\lambda_n, \dots, \lambda_{n+j})}{j!}, \quad j = 0, \dots, r-1.$$

By Lemma 5,

$$\begin{aligned} \|F\|_{L_p^r[\lambda_n, \lambda_{n+r}]}^p &\leq C_0 \left[(\lambda_{n+r} - \lambda_n)^{rp} \|F\|_{L_p^r[\lambda_n, \lambda_{n+r}]}^p + \sum_{j=0}^{r-1} (\lambda_{n+r} - \lambda_n)^{jp+1} \frac{|F(\lambda_n, \dots, \lambda_{n+j})|^p}{j!} \right] \\ &\leq C_1 \left[\|F\|_{L_p^r[\lambda_n, \lambda_{n+r}]}^p + \sum_{j=0}^{r-1} (\lambda_{n+r} - \lambda_n)^{jp+1} |F(\lambda_n, \dots, \lambda_{n+j})|^p \right], \end{aligned}$$

where $C_0 = C_0(r, p)$ comes from Lemma 5 and $C_1 = C_1(r, p, K) = C_0 \cdot \max\{1, (rK)^{rp}\}$ (see (5)). By summing over $n \in \mathbb{Z}$, using $F|_{\Lambda} = f$, and by the definition of $\|f\|_{\text{eq}, W}$, we get

$$(15) \quad \|F\|_{L^p(I)}^p \leq C_1 \left[\|F\|_{L_p^r(I)}^p + \|f\|_{\text{eq}, W}^p \right].$$

Now assume that (12) holds. Given $f \in W_p^r(I)|_{\Lambda}$, choose $F \in L_p^r(I)$ with $F|_{\Lambda} = f$ and $\|F\|_{L_p^r(I)} = \|f\|_{L_p^r(I)|_{\Lambda}}$ (recall that such a minimizer F exists because $L_p^r(I)$ is uniformly convex). Then,

$$\|F\|_{L_p^r(I)}^p = \|f\|_{L_p^r(I)|_{\Lambda}}^p \leq C\|f\|_{\text{eq}, L}^p \leq C\|f\|_{\text{eq}, W}^p,$$

by Proposition 4. Applying (15), we have

$$\|F\|_{W_p^r(I)}^p = \|F\|_{L^p(I)}^p + \|F\|_{L_p^r(I)}^p \leq (C_1 + 1)\|F\|_{L_p^r(I)}^p + C_1\|f\|_{\text{eq}, W}^p \leq (CC_1 + C + C_1)\|f\|_{\text{eq}, W}^p.$$

Hence, by definition of the trace norm, one obtains

$$\|f\|_{W_p^r(W)|_{\Lambda}}^p \leq (CC_1 + C + C_1)\|f\|_{\text{eq}, W}^p.$$

The reverse inequality comes from Proposition 4, so that this proves (a).

To prove (b), assume that $T : L_p^r(I)|_{\Lambda} \rightarrow L_p^r(I)$ is quasi-optimal and let $\|T\|$ be the norm of T as an operator $L_p^r(I)|_{\Lambda} \rightarrow L_p^r(I)$. Observe that $W_p^r(I)|_{\Lambda} \subset L_p^r(I)|_{\Lambda}$. Also, if $f \in L_p^r(I)|_{\Lambda}$, Tf is locally in W_p^r , because $Tf \in L_p^r(I)$. Hence, T will make sense as an operator taking $W_p^r(I)|_{\Lambda}$ into $W_p^r(I)$ if we can prove that $\|Tf\|_{W_p^r(I)}$ is finite whenever $f \in W_p^r(I)|_{\Lambda}$.

If $f \in W_p^r(I)|_{\Lambda}$, we use (15) and Proposition 4 to obtain

$$\begin{aligned} \|Tf\|_{W_p^r(I)}^p &= \|Tf\|_{L^p(I)}^p + \|Tf\|_{L_p^r(I)}^p \leq (C_1 + 1)\|Tf\|_{L_p^r(I)}^p + C_1\|f\|_{\text{eq}, W}^p \\ &\leq (C_1 + 1)\|T\|^p\|f\|_{L_p^r(I)|_{\Lambda}}^p + C_1C'^{-p}\|f\|_{W_p^r(I)|_{\Lambda}}^p \\ &\leq ((C_1 + 1)\|T\|^p + C_1C'^{-p})\|f\|_{W_p^r(I)|_{\Lambda}}^p. \end{aligned}$$

Here we have used $\|f\|_{L_p^r(I)|_{\Lambda}} \leq \|f\|_{W_p^r(I)|_{\Lambda}}$, which is trivial. This shows that $T : W_p^r(I)|_{\Lambda} \rightarrow W_p^r(I)$ is quasi-optimal. \square

Now we can give the proof of Theorem 1.

Proof of Theorem 1. Observe that for any $f : \Lambda \rightarrow \mathbb{C}$, the function $\Phi_r f$ is locally in L_p^r , because it is of class \mathcal{C}^{r-1} and piecewise \mathcal{C}^r . Hence, to check that $\Phi_r(L_p^r(I)|_{\Lambda}) \subset L_p^r(I)$, it is enough to see that $\|\Phi_r f\|_{L_p^r(I)}$ is finite for any $f \in L_p^r(I)|_{\Lambda}$.

Therefore, we only need to prove that

$$(16) \quad \|\Phi_r f\|_{L_p^r(I)} \leq C(r, p)\|f\|_{\text{eq}, L}, \quad r = 1, 2.$$

Then we will have

$$\|f\|_{L_p^r(I)|_{\Lambda}} \leq \|\Phi_r f\|_{L_p^r(I)} \leq C(r, p)\|f\|_{\text{eq}, L} \leq C'(r, p)\|f\|_{L_p^r(I)|_{\Lambda}},$$

where the last inequality comes from Proposition 4, so (a) will follow. Using Theorem 2, (b) follows from (a), because the fact that

$$\|\Phi_r\|_{W_p^r(I)|_{\Lambda} \rightarrow W_p^r(I)} \leq C'(r, p, K), \quad r = 1, 2$$

is contained in its proof.

Inequality (16) will follow from some easy computations using the formulas given in Section 2. The analogous estimate for $W_p^r(I)$ can also be obtained directly by the same means instead of appealing to Theorem 2.

We first deal with the case $r = 1$. We compute $\|\Phi_1 f\|_{L_p^1[\lambda_n, \lambda_{n+1}]}$ using formula (8):

$$\|\Phi_1 f\|_{L_p^1[\lambda_n, \lambda_{n+1}]}^p = |f(\lambda_n, \lambda_{n+1})|^p \|1\|_{L^p[\lambda_n, \lambda_{n+1}]}^p = (\lambda_{n+1} - \lambda_n) |f(\lambda_n, \lambda_{n+1})|^p.$$

By summing over $n \in \mathbb{Z}$, we obtain

$$\|\Phi_1 f\|_{L_p^1(I)}^p = \|f\|_{\text{eq}, L}^p.$$

For the case $r = 2$, we compute $\|\Phi_2 f\|_{L_p^2[\lambda_n, \mu_n]}^p$ using formula (11). We have, for $x \in [\lambda_n, \mu_n]$,

$$(\Phi_2 f)''(x) = f(\lambda_{n-1}, \lambda_n, \lambda_{n+1}) q''\left(\frac{x - \lambda_n}{h_n}\right).$$

Making the change of variables $y = (x - \lambda_n)/h_n$, we compute

$$\left\| q''\left(\frac{x - \lambda_n}{h_n}\right) \right\|_{L^p[\lambda_n, \mu_n]}^p = h_n \|q''\|_{L^p[0, 1/2]}^p = M h_n,$$

where $M = M(p)$ is some constant depending only on p .

Hence,

$$(17) \quad \|\Phi_2 f\|_{L_p^2[\lambda_n, \mu_n]}^p = M h_n |f(\lambda_{n-1}, \lambda_n, \lambda_{n+1})|^p.$$

Proceeding analogously, we obtain

$$(18) \quad \|\Phi_2 f\|_{L_p^2[\mu_{n-1}, \lambda_n]}^p = M h_{n-1} |f(\lambda_{n-1}, \lambda_n, \lambda_{n+1})|^p.$$

By summing inequalities (17) and (18) over $n \in \mathbb{Z}$ and using $h_{n-1} + h_n = \lambda_{n+1} - \lambda_{n-1}$, we find that

$$\|\Phi_2 f\|_{L_p^2(I)}^p = M \|f\|_{\text{eq}, L}^p.$$

This proves the Theorem. \square

5. SIMPLIFICATION OF THE EXPRESSION FOR $\|f\|_{\text{eq}, W}$

In this Section, we examine whether the expression for $\|f\|_{\text{eq}, W}$ given in (3) can be simplified by eliminating the terms corresponding to $1 \leq j \leq r-1$. Throughout the Section we will assume that (5) holds.

The next proposition proves that the answer is affirmative for $r = 2$. Its proof reduces to doing some easy but somewhat lengthy estimates on the divided differences.

Proposition 6. *Let $r = 2$ and assume that (5) holds. Define $\|f\|_{\text{eq}, W}$ and $\|f\|_{\text{simp}, W}$ from (3) and (7). Then we have the equivalence of norms*

$$\|f\|_{\text{eq}, W} \approx \|f\|_{\text{simp}, W}.$$

The equivalence constants depend only on p, K .

Proof. First, we have

$$|f(\lambda_n)| \leq |f(\lambda_{n-1})| + (\lambda_{n+1} - \lambda_{n-1}) |f(\lambda_{n-1}, \lambda_n)|.$$

Hence,

$$(\lambda_{n+1} - \lambda_{n-1}) |f(\lambda_n)|^p \leq 2^{p-1} [(\lambda_{n+1} - \lambda_{n-1}) |f(\lambda_{n-1})|^p + (\lambda_{n+1} - \lambda_{n-1})^{p+1} |f(\lambda_{n-1}, \lambda_n)|],$$

by Hölder's inequality for sums. Adding over $n \in \mathbb{Z}$, we obtain $\|f\|_{\text{simp}, W}^p \leq 2^{p-1} \|f\|_{\text{eq}, W}^p$.

To prove the reverse inequality, we fix $n \in \mathbb{Z}$. Since $\lambda_{n+2} - \lambda_n = (\lambda_{n+2} - \lambda_{n+1}) + (\lambda_{n+1} - \lambda_n)$, we have either $\lambda_{n+2} - \lambda_{n+1} \geq (\lambda_{n+2} - \lambda_n)/2$ or $\lambda_{n+1} - \lambda_n \geq (\lambda_{n+2} - \lambda_n)/2$. Assume that $\lambda_{n+2} - \lambda_{n+1} \geq (\lambda_{n+2} - \lambda_n)/2$. The other case is similar and easier.

By the triangle inequality and the inequality $|x + y|^p \leq 2^{p-1} [|x|^p + |y|^p]$, one obtains the following inequalities:

$$(19) \quad |f(\lambda_n, \lambda_{n+1})|^p \leq 2^{p-1} [(\lambda_{n+2} - \lambda_n)^p |f(\lambda_n, \lambda_{n+1}, \lambda_{n+2})|^p + |f(\lambda_{n+1}, \lambda_{n+2})|^p],$$

$$(20) \quad (\lambda_{n+2} - \lambda_n)^p |f(\lambda_{n+1}, \lambda_{n+2})|^p \leq 2^{p-1} [|f(\lambda_{n+1})|^p + |f(\lambda_{n+2})|^p],$$

$$(21) \quad (\lambda_{n+2} - \lambda_n) |f(\lambda_{n+2})|^p \leq 2(\lambda_{n+2} - \lambda_{n+1}) |f(\lambda_{n+2})|^p.$$

By applying inequalities (19), (20), (21), one gets

$$(22) \quad \begin{aligned} (\lambda_{n+2} - \lambda_n)^{p+1} |f(\lambda_n, \lambda_{n+1})|^p &\leq 2^{2p-2} (\lambda_{n+2} - \lambda_n) |f(\lambda_{n+1})|^p \\ &+ 2^{2p-1} (\lambda_{n+3} - \lambda_{n+1}) |f(\lambda_{n+2})|^p + 2^{p-1} (\lambda_{n+2} - \lambda_n)^{2p+1} |f(\lambda_n, \lambda_{n+1}, \lambda_{n+2})|^p. \end{aligned}$$

Hence, we have

$$\begin{aligned}
 (23) \quad & (\lambda_{n+2} - \lambda_n)|f(\lambda_n)|^p + (\lambda_{n+2} - \lambda_n)^{p+1}|f(\lambda_n, \lambda_{n+1})|^p \leq \\
 & \leq 2^{p-1}(\lambda_{n+2} - \lambda_n)|f(\lambda_{n+1})|^p + (2^{p-1} + 1)(\lambda_{n+2} - \lambda_n)^{p+1}|f(\lambda_n, \lambda_{n+1})|^p \\
 & \leq C(p, K) \left[(\lambda_{n+2} - \lambda_n)|f(\lambda_{n+1})|^p + (\lambda_{n+3} - \lambda_{n+1})|f(\lambda_{n+2})|^p \right. \\
 & \quad \left. + (\lambda_{n+2} - \lambda_n)|f(\lambda_n, \lambda_{n+1}, \lambda_{n+2})|^p \right],
 \end{aligned}$$

where in the last inequality we have used (22) and (5).

In the case $\lambda_{n+1} - \lambda_n \geq (\lambda_{n+2} - \lambda_n)/2$, one obtains

$$\begin{aligned}
 (24) \quad & (\lambda_{n+2} - \lambda_n)|f(\lambda_n)|^p + (\lambda_{n+2} - \lambda_n)^{p+1}|f(\lambda_n, \lambda_{n+1})|^p \leq \\
 & \leq C'(p)[(\lambda_{n+1} - \lambda_{n-1})|f(\lambda_n)|^p + (\lambda_{n+2} - \lambda_n)|f(\lambda_{n+1})|^p].
 \end{aligned}$$

In fact, (5) is not necessary in this case.

For every index $n \in \mathbb{Z}$, we have either (23) or (24). By summing these inequalities over $n \in \mathbb{Z}$, we obtain $\|f\|_{\text{eq},W}^p \leq C''(p, K)\|f\|_{\text{simp},W}^p$. \square

Now we give an example that shows that, in general, for any $r \geq 3$, one cannot eliminate all the terms $1 \leq j \leq r-1$ from the expression of $\|f\|_{\text{eq},W}$. Assume that $|I| < \infty$ and suppose that one wishes to prove the following equivalence of norms:

$$(25) \quad \|f\|_{W_p^r(I)|_\Lambda}^p \approx \|f\|_{\text{eq},L}^p + \sum_{n \in \mathbb{Z}} c_n |f(\lambda_n)|^p,$$

for some coefficients $\{c_n\}$ not depending on f , and with equivalence constants depending only on r, p, K . Taking $f \equiv 1$ we see that the coefficients $\{c_n\}$ must be in ℓ^1 and satisfy $\|\{c_n\}\|_1 \leq C(r, p, K)|I|$. Then, the next example shows that (25) cannot hold in some cases.

Example 1. Let $r \geq 3$. Fix a small $h > 0$ and let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be a strictly increasing sequence such that $\lambda_0 = -1$, $\lambda_1 = 1$, $\lambda_n \rightarrow 1 + h$ as $n \rightarrow +\infty$ and $\lambda_n \rightarrow -1 - h$ as $n \rightarrow -\infty$. This sequence Λ satisfies (5) with $K = 2$.

Put $p(x) = [(1+h)^2 - x^2]/[h(2+h)]$ and consider the interpolation problem given by $f(\lambda_n) = p(\lambda_n)$. Then $F(x) = p(x)$ solves this problem. Now observe that $\|p\|_{L_p^r(I)} = 0$, $\|p\|_{W_p^r(I)} \approx 1/h$ and $|p(\lambda_n)| \leq 1$ for all $n \in \mathbb{Z}$. Hence, the right hand side of (25) is less or equal than $C(r, p, K)|I|$.

It follows from (15) and Proposition 4 that

$$\|p\|_{W_p^r(I)}^p \leq C'(r, p, K)\|f\|_{\text{eq},W}^p \leq C''(r, p, K)\|f\|_{W_p^r(I)|_\Lambda}^p.$$

Hence, the left hand side of (25) is greater or equal than $C''(r, p, K)^{-1}/h^p$. This is a contradiction for small enough $h > 0$.

However, the former example is in some sense pathological because it does not satisfy the property that one can choose $r+1$ points λ_n which are “well separated”. It turns out that this kind of condition is the only thing one needs to be able to eliminate the terms $1 \leq j \leq r-1$ in the expression for $\|f\|_{\text{eq},W}$. This always happens if the interval I is large enough. In fact, one can prove the following Proposition.

Proposition 7. Let $r \geq 1$, $1 < p < \infty$ and assume that $|I| \geq (4r+2)K$ (see (5)). Define $\|f\|_{\text{simp},W}$ from (7). Then Theorem 2 remains true if one replaces $\|f\|_{\text{eq},W}$ with $\|f\|_{\text{simp},W}$.

We will only give a sketch of the proof.

First one can prove that if J is an interval with $|J| = (4r+2)K$, and $\eta_k \in J$, $k = 1, \dots, r+1$ satisfy $|\eta_k - \eta_l| \geq K$ for $k \neq l$, then

$$\|F\|_{L_p^r(J)}^p \leq C(r, p, K) \left[\|F\|_{L_p^r(J)}^p + \sum_{k=1}^{r+1} |F(\eta_k)|^p \right].$$

Now one covers I with intervals I_j of length $(4r+2)K$ in such a way that the interiors of any three of these intervals do not intersect. One obtains the above inequality for $J = I_j$.

By averaging the above estimate among all possible choices of $\eta_k = \lambda_{n_k} \in \Lambda \cap I_j$ satisfying $|\eta_k - \eta_l| \geq K$ for $k \neq l$ and summing over all indices j , one gets

$$\|F\|_{L^p(J)}^p \leq C \left[\|F\|_{L_p^r(J)}^p + \sum_{n \in \mathbb{Z}} (\lambda_{n+1} - \lambda_{n-1}) |F(\lambda_n)|^p \right] \leq C \left[\|F\|_{L_p^r(J)}^p + \|f\|_{\text{simp},W} \right],$$

where $C = C(r, p, K)$ and $f = F|_\lambda$.

Now one can repeat the arguments of the proof of (a) in Theorem 2, replacing $\|f\|_{\text{eq},W}$ by $\|f\|_{\text{simp},W}$ and using this last inequality instead of (15). Here, one must use the inequalities $\|f\|_{\text{eq},L} \leq \|f\|_{\text{simp},W}$, which is trivial, and $\|f\|_{W_p^r(I) \cap \Lambda} \geq C' \|f\|_{\text{simp},W}$, which is proved in the same way as (b) in Proposition 4. Then, it is easy to see that all the steps made after (15) in the proof of (a) in Theorem 2 remain true when we replace $\|f\|_{\text{eq},W}$ with $\|f\|_{\text{simp},W}$.

6. OPEN QUESTIONS

The first open questions are whether one can generalize our results on \mathbb{R} to a higher number of derivatives. In particular, we ask the following.

Question 1. Are (4) and (6) true for $r \geq 3$?

Theorem 2 shows that if (4) is true for some values of r, p , then (6) is also true for the same values of r, p .

Question 2. Suppose that the answer to Question 1 is affirmative. Can one give explicit quasi-optimal spline interpolators for $r \geq 3$?

As before, Theorem 2 shows that it is enough to prove that some interpolator is quasi-optimal for L_p^r .

Another matter is whether one can give similar results in higher dimension, i.e. on \mathbb{R}^d , $d \geq 2$. Fefferman, Israel and Luli show in [9] that it is impossible to obtain bounded depth interpolators for $d \geq 2$. Hence, one cannot obtain simple interpolators like Φ_1 and Φ_2 for a general configuration of points.

However, it might be possible to obtain nice formulas and explicit interpolators if one imposes some kind of regularity conditions on the configuration of points. The conditions should be mild enough to allow the usual cases which appear in applications.

One possible condition is the minimum angle condition, which is widely used in numerical analysis. It means that in the triangular mesh formed by the points, the interior angles of every triangle are uniformly bounded from below. See [2] for a review on this and other usual geometric conditions in numerical analysis.

Hence, we ask the following question.

Question 3. Can one give simple formulas for the trace space norm and quasi-optimal explicit interpolators in $L_p^r(\mathbb{R}^d)$ and $W_p^r(\mathbb{R}^d)$ if one imposes some regularity conditions on the allowed configurations of points?

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